Maximum distance attained by a ballistic projectile launched from a free swinging rigid pendulum *

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Contents

*When to spit a cherry pip from a swing

List of Figures

1 List of symbols

Symbol	**Description**	**Units**
a, b, c	coefficients of a general quadratic equation	
E	energy (kinetic + potential)	J
E_K	kinetic energy	J
E_P	potential energy	J
F_y	vertical force due to gravity during ballistic flight	N
g	gravitational acceleration	m/s^2
h_0	rest height of pendulum	m
L	length of pendulum	m
λ	root of a general quadratic equation	
m	mass of projectile	kg
θ	angle of pendulum from vertical	rad
θ_R	angle of pendulum from vertical at release	rad
θ_{max}	maximum angle of pendulum from vertical	rad
s_x	horizontal distance covered by projectile	m
t	time since release	s
t_i	time at impact	s
t_R	time at release	s
V	speed of projectile	m/s^2
V^+	additional velocity imparted to projectile at release	m/s
V_R	total speed of projectile at release	m/s^2
x	horizontal position	m
x_R	horizontal position at release	m
$\dot{x}$	horizontal velocity	m/s
$\dot{x}_R$	horizontal velocity at release	m/s
y	vertical position	m
y_R	vertical position at release	m
$\dot{y}$	vertical velocity	m/s
$\dot{y}_R$	vertical velocity at release	m/s

Part I
Introduction

2 Problem definition: Long Shot

This problem was defined in "The Last Word" of New Scientist,

> "My five-year old daughter wants to know if you spit a cherry stone while swinging on a swing, would it go farthest if you spit it while you are at the lowest point of the swing (because you are moving so fast), or would it be better to spit it at the highest point?"

3 Assumptions

1. The swing is rigid in tension (i.e. it will trace a circular arc)
2. Frictional effects are neglected
3. The swing is not being pushed at the time of release
4. The projectile will be released tangentially to the motion of the swing
5. Air resistance and wind are neglected
6. The ground is flat and there are no obstructions in the flight path
7. The projectile will not bounce on impact

4 Flight profile

There are three distinct phases in the flight of our projectile.

1. For the first part of the flight, the projectile is constrained to follow a circular trajectory (inside the head of the child) and can be considered in a similar manner to that of a pendulum of length L where L is the distance between the pivot of the swing and the child's head. The motion of the projectile may be described by the equations of "Simple Harmonic Motion".

2. The second phase of the flight is the release. It can be assumed that the projectile is ejected tangentially to the motion of the swing and that a constant additional velocity V_R will be imparted to it, regardless of the angle of the swing at the time of release.

3. The final phase of the flight is ballistic, similar to a thrown ball. Neglecting air resistance and wind, the only force acting on the projectile in this phase is the gravitational attraction of the Earth.

Part II
Equations of motion

5 Simple Harmonic Motion

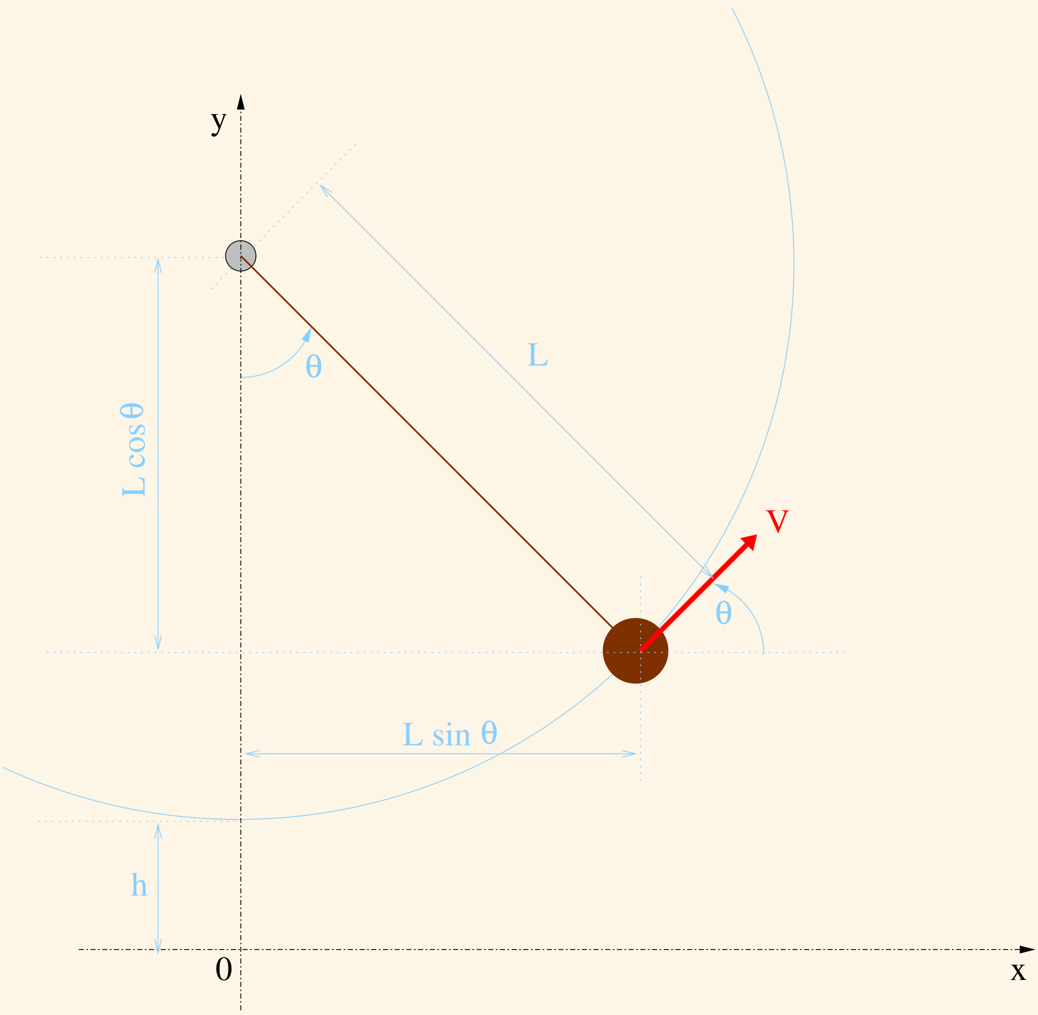

Figure 1: Pendulum (Simple Harmonic Motion)

The position of the projectile while constrained to follow the circular arc of a pendulum is

$$\begin{aligned} x(\theta) &= L\ sin(\theta) \\ y(\theta) &= L\ (1 - cos(\theta)) + h_0 \end{aligned} \tag{1}$$

where x is the horizontal position, y is the vertical position, L is the distance from the pivot, θ is the angle of the pendulum measured from the vertical axis and h_0 is the height above the ground of the projectile when the pendulum is vertical.

The velocity at any point in the trajectory can be determined by the equation

$$V(\theta) = \sqrt{2\ g\ L\ \ (cos(\theta) - cos(\theta_{max}))} \tag{2}$$

where g is the acceleration due to gravity and θ_{max} is the maximum angle that the pendulum makes with the vertical axis, i.e. the angle at which it reverses direction.

5.1 Derivation of particle velocity

The velocity of the particle may be determined from consideration of its energy. The particle's potential energy E_P is

$$E_P(\theta) = m\ g\ y(\theta) \tag{3}$$

and its kinetic energy E_K is

$$E_K(\theta) = \frac{1}{2}\ m\ V^2(\theta) \tag{4}$$

where m is the mass of the particle, g is the gravitational attraction due to the mass of the Earth ($g \approx 9.81$m/s), V is the speed of the particle (velocity tangential to its motion) and y its height above the ground.

The total energy of the particle at any point in its trajectory is equal to sum of its potential energy and its kinetic energy

$$\begin{aligned} E(\theta) &= E_K(\theta) + E_P(\theta) \\ &= \frac{1}{2}\ m\ V^2(\theta) + m\ g\ y(\theta) \end{aligned} \tag{5}$$

Substituting for $y(\theta)$ from equation (1), the energy can be expressed as

$$E(\theta) = \frac{1}{2}\ m\ V^2(\theta) + m\ g\ \ (L\ (1 - cos(\theta)) + h_0) \tag{6}$$

Now, we know that at its maximum displacement, the swing's velocity is zero, i.e.

$$E(\theta_{max}) = m\ g\ (h + L\ (1 - cos(\theta_{max}))) + 0 \tag{7}$$

Conservation of energy means that in the absence of friction,

$$E(\theta) = E(\theta_{max}) \tag{8}$$

We can therefore equate the right hand sides of equations (6) and (7)

$$m\ g\ (h + L\ (1 - cos(\theta))) + \frac{1}{2}\ m\ V^2(\theta) = m\ g\ (h + L\ (1 - cos(\theta_{max}))) + 0 \quad (9)$$

Dividing both sides by m and re-arranging gives the speed V as a function of angle θ

$$\boxed{V(\theta) = \sqrt{2\ g\ L\ (cos(\theta) - cos(\theta_{max}))}}$$

6 Release

At the time of release, the particle is moving with velocity $V(\theta_R)$ (governed by equation (2)) at a position (x_R, y_R). The speed of the projectile immediately after release V_R is

$$V_R = V(\theta_R) + V^+ \quad (10)$$

where V^+ is the additional velocity imparted during ejection. It is assumed that this velocity is in the same direction as $V(\theta_R)$. The particle's position immediately after release (x_R,y_R) is

$$\begin{aligned} x_R &= L\ sin(\theta_R) \\ y_R &= L\ (1 - cos(\theta_R)) + h_0 \end{aligned} \quad (11)$$

The velocity may be resolved into components parallel with the x and y axis

$$\begin{aligned} \dot{x}_R &= (V_R + V^+)\ cos(\theta_R) \\ \dot{y}_R &= (V_R + V^+)\ sin(\theta_R) \end{aligned} \quad (12)$$

7 Ballistic Motion

The only force acting on the projectile during its ballistic phase of flight is the force of gravity. There is no horizontal component to this gravitational force so the x-component of the particle velocity will remain constant until the particle hits the ground

$$\dot{x}(t) = \dot{x}_R \quad (13)$$

The total distance covered by the particle s_x will be given by the distance covered while airborne, equal to the x-component of velocity $\dot{x}$ times the length of time for which it is airborne t, added to its position at the time of

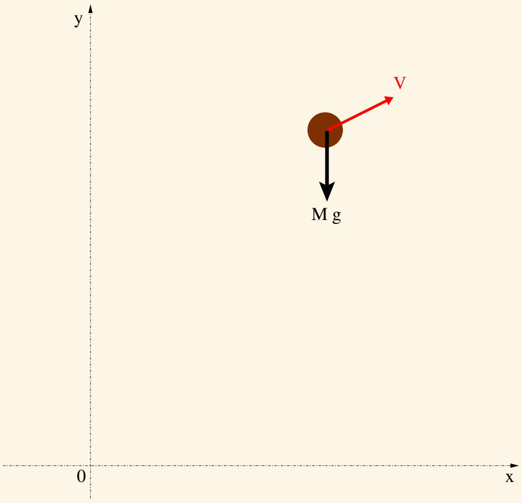

Figure 2: Ballistic Motion

release

$$\begin{aligned} s_x &= x_R + \int_0^{t_i} \dot{x}(t)\ dt \\ &= x_R + \int_0^{t_i} \dot{x}_R\ dt \\ &= x_R + \dot{x}_R\ t_i \end{aligned} \tag{14}$$

The length of time for which the projectile is airborne may be determined from consideration of its vertical motion. Its vertical acceleration $\ddot{y}$ is of equal magnitude to the acceleration due to gravity

$$\ddot{y} = -g \tag{15}$$

The vertical velocity $\dot{y}$ at any time t is then given by

$$\begin{aligned} \dot{y}(t) &= \dot{y}_R + \int_0^{t} -g\ dt \\ &= \dot{y}_R - g\ t \end{aligned} \tag{16}$$

Integrating this again with respect to time gives the particle's position at any time t after release

$$\begin{aligned} y(t) &= y_R + \int_0^{t} (\dot{y}_R - g\ t)\, dt \\ &= y_R + \dot{y}_R\ t - \frac{1}{2}\ g\ t^2 \end{aligned} \tag{17}$$

Now, if the ground is level, we know that at the time of impact t_i the vertical position of the particle must be zero ($y(t_i) = 0$). Substituting this into the equation for the particle's velocity gives a quadratic equation in t_i which can be solved

$$y(t_i) = y_R + \dot{y}_R \; t_i - \frac{1}{2} \, g \, t^2 = 0 \tag{18}$$

Solving the quadratic using

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4 \, a \, c}}{2 \, a} \qquad \text{(General solution to quadratic equation)}$$

gives two possible solutions

$$\begin{aligned} \lambda_1 &= \frac{\dot{y}_R}{g} + \frac{\sqrt{\dot{y}_R^2 + 2 \, g \, y_R}}{g} \\ \lambda_2 &= \frac{\dot{y}_R}{g} - \frac{\sqrt{\dot{y}_R^2 + 2 \, g \, y_R}}{g} \end{aligned} \tag{19}$$

Assuming that the projectile was released with positive y and $\dot{y}$, the first solution λ_1 is clearly the one to use. λ_2 will be negative and refers to the time at which the projectile would have been launched if it had been fired on a ballistic trajectory from the ground level.

$$t_i = \lambda_1 = \frac{\dot{y}_R}{g} + \frac{\sqrt{\dot{y}_R^2 + 2 \, g \, y_R}}{g} \tag{20}$$

Part III
Results

8 Distance as a function of release angle

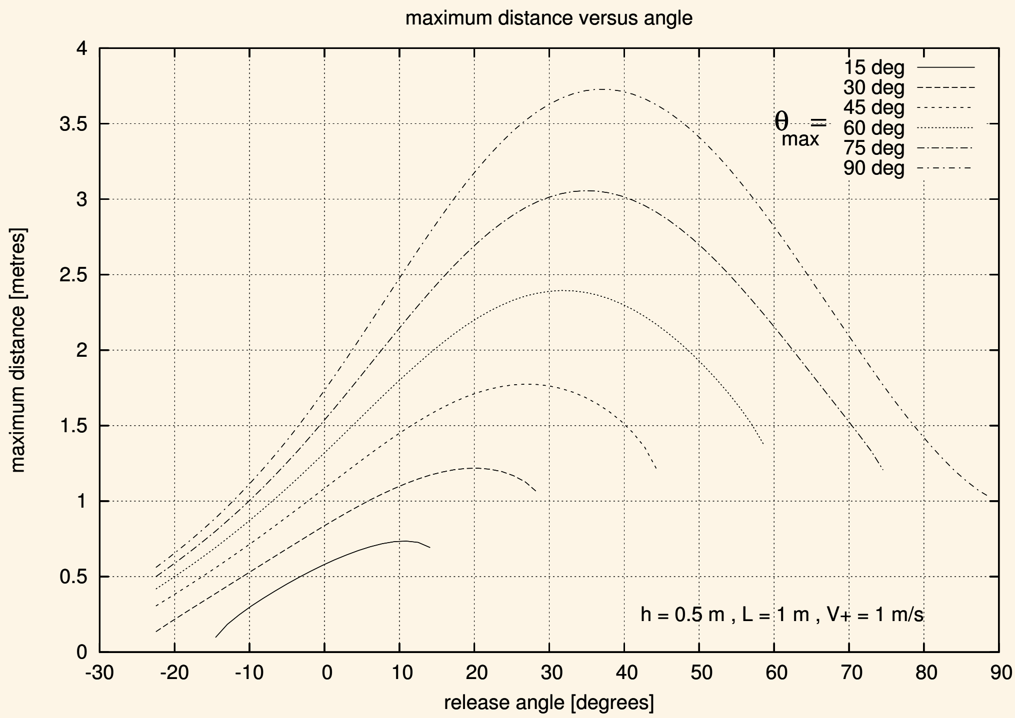

Figure 3: Maximum distance as a function of release angle

A Plotting the results

A GNU Octave[1] script was created to plot the results. GNU Octave is available from http://www.octave.org

```
#! /usr/bin/octave -f
##  -*-octave-*-

parameters.g = 9.81;   # gravitational acceleration [m/s^2]
parameters.h = 0.5;   # height of pip above ground [m]
parameters.L = 1.0;   # distance from pivot to pip [m]
parameters.theta_max = 2*pi/3;   # maximum angle of pendulum [rad]
parameters.theta_rel = pi/2;   # release angle of pendulum [rad]
parameters.Vplus = 1.0;   # extra velocity at release [m/s]

##############################################################
function [x, y, xdot, ydot] = pendulum (parameters)

  ## position and velocity of a pendulum for a given angle from vertical
  ##
  ## x [m] distance along x axis (horizontal)
  ## y [m] distance along y axis (vertical)
  ## v_x [m/s] velocity along x axis
  ## v_y [m/s] velocity along y axis

  g = parameters.g;
  h = parameters.h;
  L = parameters.L;
  theta_max = parameters.theta_max;
  theta_rel = parameters.theta_rel;

  ## tangential velocity of an ideal pendulum
  v_t = sqrt (2*g*L*(cos(theta_rel)-cos(theta_max)));

  ## x and y co-ordinates [m]
  x = L * sin(theta_rel);
  y = L * (1 - cos(theta_rel)) + h;

  ## x and y velocities [m/s]
  xdot = v_t * cos(theta_rel);
  ydot = v_t * sin(theta_rel);
endfunction
##############################################################
```

```
function s_x = flight (parameters)
  g = parameters.g;  [xr, yr, xdot, ydot] = pendulum (parameters);
  xdotr = xdot ...
      + parameters.Vplus * parameters.L * cos(parameters.theta_rel);
  ydotr = ydot ...
      + parameters.Vplus * parameters.L * sin(parameters.theta_rel);
  ti = ydotr / parameters.g ...
      + sqrt(ydotr^2 + 2*parameters.g*yr) / parameters.g;
  s_x = xr + xdotr * ti;
endfunction
#############################################################
function plot_results (parameters)
  for j = [1:6]
    i = 0;
    parameters.theta_max = j * pi/12;
    for theta = linspace (-pi/8, 3*pi/4, 100)
      i++;
      parameters.theta_rel = theta;
      data(i, 1  ) = theta;
      if (abs(theta) < parameters.theta_max)
data(i, 1+j) = flight (parameters);
      else
data(i, 1+j) = 0;
      endif
      if (data(i, 1+j) <= 0)
data(i, 1+j) = nan; # don't plot y=0
      endif
    endfor;
  endfor;
  oneplot;
  gset title "maximum distance versus angle"
  gset xlabel "release angle [degrees]"
  gset ylabel "maximum distance [metres]"
  gset key
  gset grid
  gset xtics 10

  eval(sprintf("gset label \"h = %g m , L = %g m , V+ = %g m/s\" at \
  80,0.25 right ", parameters.h, parameters.L, parameters.Vplus));

  gset label "q   =" at 60,3.5 left font "Symbol,20"
  gset label "max" at 61,3.4
  plot (data(:,1)*180/pi, data(:,2), '-;15 deg;',
```

```
data(:,1)*180/pi, data(:,3), '-;30 deg;',
data(:,1)*180/pi, data(:,4), '-;45 deg;',
data(:,1)*180/pi, data(:,5), '-;60 deg;',
data(:,1)*180/pi, data(:,6), '-;75 deg;',
data(:,1)*180/pi, data(:,7), '-;90 deg;')
  gset output "results.eps";
  gset terminal postscript eps;
  replot;
endfunction
#############################################################
function test_pendulum (parameters)
  n = 101;
  theta = linspace (-parameters.theta_max, parameters.theta_max, n);
  for i = 1 : n
    parameters.theta_rel = theta(i);
    [x(i), y(i), xdot(i), ydot(i)] = pendulum (parameters);
  endfor;

  v_t = sqrt(xdot.^2 .+ ydot.^2);

  data = [x, y, v_t];

  gset autoscale x
  gset autoscale y
  gset nokey
  gset grid xtics
  multiplot(2,2)
  subwindow(1,1)
  gset title "speed versus x and y position"
  gset grid xtics ytics ztics
  gset xlabel 'x [m]'
  gset ylabel 'y [m]'
  gset zlabel 'speed [m/s]'
  gset view 45 , 45
  gset parametric
  gsplot data title 'pendulum' with points
  gset noparametric
  subwindow(2,1)
  gset title "speed versus angle"
  gset polar
  gset grid polar
  gset size square
  polar_data = [theta', v_t];
```

```
  gset xrange [-8:8]
  gset yrange [-8:8]
  gplot polar_data with points
  gset nopolar
  gset grid nopolar
  gset size nosquare
  gset autoscale x
  gset autoscale y
  subwindow(1,2)
  gset title "y position versus x position"
  gset grid xtics ytics
  gset xlabel 'x position [m]'
  gset ylabel 'y position [m]'
  plot (x, y)
  subwindow(2,2)
  gset title "speed versus x position"
  gset grid xtics ytics
  gset xlabel 'x position [m]'
  gset ylabel 'speed [m/s]'
  plot (x, v_t);
endfunction
#############################################################
## test_pendulum (parameters); pause;
plot_results (parameters);
```

References

[1] John W. Eaton. *GNU Octave Manual.* Network Theory Limited, 2002.